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PLATONIC MECHANISM DESIGN

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Abstract

We characterize the class of tiered exchange functions for an assignment problem. We examine a model with a finite number of indivisible goods to be assigned to a finite number of individuals with status quo endowments. However, these individuals can be partitioned into tiers, and new axioms of social justice are developed to account for this tiering.

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1 Introduction

In this paper, we determine a complete characterization of the set of Pareto consistent, nonbossy, and strategyproof mechanisms that satisfy axioms of tiered social justice. We examine a model with a finite number of indivisible goods to be assigned to a finite number of individuals with status quo endowments. These individuals are partitioned into tiers (or classes) of more and less privileged agents. We develop new axioms of social equity to account for this tiering.

A great deal has been written on assignment games with one of two structures. Either all agents are on a level playing field, or they are fully ordered into a hierarchy. There seems to be a gap, however, in between the two extremes. In such environments, agents within a stratum are on a level playing field, but have rights which supercede the rights of agents in strata below. In environments such as an academic department, a firm, or a fraternity, a tiered system predominates. That is, all agents can be assigned to one of many tiers that partition the set of agents. For example, in a university department, all agents can be classified as full professors, associate professors, assistant professors, dirt, and graduate students, in that order.

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Within this environment, we need to develop new axioms of social justice. For inspiration on how to develop these, we look towards the Platonic ideal, as described in *The Republic*, Book IV [2], where Plato describes why we should afford different luxuries to different classes of citizens:

> And so I say to you, do not compel us to assign to the guardians a sort of happiness which will make them anything but guardians; ...Our potters also might be allowed to repose on couches, and feast by the fireside, passing round the winecup, while their wheel is conveniently at hand, and working at pottery only as much as they like; in this way we might make every class happy-and then, as you imagine, the whole State would be happy. But do not put this idea into our heads; for, if we listen to you, ...the potter will cease to be a potter, and no one will have the character of any distinct class in the State... And therefore we must consider whether in appointing our guardians we would look to their greatest happiness individually, or whether this principle of happiness does not rather reside in the State as a whole. But the latter be the truth, then the guardians and auxillaries, and all others equally with them, must be compelled or induced to do their own work in the best way. And thus the whole State will grow up in a noble order...

We interpret the Platonic ideal in the framework of a mechanism design problem in the following manner. Individual rationality no longer makes sense in its traditional form. That is, if a member of a higher tier prefers a lower-tiered individual's endowment, he should be able to take it. However, some sense of property rights for this lowly individual is called for, and we develop this new definition. The same sort of argument can be made for a new definition of envy-freeness. Once these are established, we turn to issues of characterization and implementation, for which we propose a mechanism that implements an allocation rule in dominant strategies, subject to the new axioms of social justice we develop and some desirable traditional ones.

In the formative study of models with indivisibilities, Shapley and Scarf [7] established there always exists a non-empty core. They introduce the idea of a top trading cycle, suggested by Gale, whose outcome is always in the core, and for which competitive prices exist. Roth and Postlewaite [6] established that the core defined by weak domination is always non-empty, and contains exactly one allocation. Roth [5] then showed that the core could be implemented by a strategyproof allocation mechanism. Bird [1] showed that this mechanism is also coalitionally strategyproof.

Papai [4] provides a complete characterization of the set of Pareto optimal[1], nonbossy and strategyproof mechanisms, and this class is named hierarchical exchange functions, of which the top trading cycle is a member. The mechanisms characterized in this paper are a sub-class of these mechanisms. Additionally, Svensson [8] explores the characterization of strategyproof mechanisms under the assumptions of individual rationality and Pareto consistency, and provides an alternative proof of Ma [3] that a mechanism satisfies these restrictions if and only if it is the core mechanism.

2 Model and Definitions

2.1 Without Tiers

First, we describe the environment and define our terms in an environment without tiers. Let $N = \{1, ...n\}$ denote the set of agents, and $K = \{1, ..., k\}$ denote the set of indivisible goods, such that $K \geq N \geq 2$. Let R_i summarize agent i's preferences over the set K. Let R_i denote the set of admissible preferences for agent i, so that $R_i \in R_i$, for all $i \in N$. We assume that preferences are strict and that each object is preferred to nothing, and that no agent has use for more than one good. Let P_i be the denotation for strict preferences, and P_i denote the set of admissible strict preferences. Let $P = \times_{i \in N} P_i$.

A status quo endowment is an ordered list $E_0 = (E_0(1), ..., E_0(n))$ for the n agents, such that $E_0(i)$ is the is the object agent i possesses before we begin the exchange, $E_0(i) \in K \backslash \emptyset$. An allocation is an ordered list $x = (x_1, ..., x_n)$ for the n agents, such that x_i is the object assigned to agent i, $x_i \in K \backslash \emptyset$ for all i. Note that a status quo endowment is feasible if for all $i, j \in N, E_0(i) \neq E_0(j)$ and $|E_0(i)|$ 1 for all $i \in N$ and an allocation is feasible if for all $i, j \in N, x_i \neq x_j$ and if $|x_i|$ 1 for all $i \in N$. Denote the set of feasible allocations by X. For a given preference profile, $P \in P$, an allocation is said

[1] *Pareto consistent* in our terminology

to be Pareto efficient if there does not exist an $x' \in X$ such that for all $i \in N$, $x_i' R_i x_i$ and, for some $j \in N, x_j' P_j x_j$. An allocation is said to be individually rational if for all $i \in N, x_i R_i E_0(i)$. An allocation is envy-free if for all $i, j \in N, x_i R_i x_j$.

A social choice function (SCF) is a mapping $f : P \to X$. Let $f_i(P) = x_i$ denote the allocation to agent i by f at preference profile P. The SCF is said to be strategyproof if for all $i \in N$ and all $P \in P, f_i(P) P_i f_i(P_{-i}, P_i')$. If f is not strategyproof, it is said to be manipulable. A SCF is said to be Pareto consistent if $f(P) = x$ is Pareto efficient for all $P \in P$. A SCF is said to be nonbossy if for all $P \in P$, all $i, j \in N$, and all $P_i' \in P_i$, if $f_i(P) = f_i(P_{-i}, P_i')$, then $f_j(P) = f_j(P_{-i}, P_i')$. A SCF is coalitionally strategyproof if for all $C \subseteq N$, all $P \in P$, and all $P_C' \in P_C$, there exists an $i \in C$ such that $f_i(P) P_i f_i(P_C', P_{-C})$.

2.2 With Tiers – The Axioms of Tiered Social Justice

The focus of this analysis is environments with tiers, where some agents have rights that obscure those of others. The definitions in section 2.1 describe environments where all agents may have property rights over their status quo endowments of the indivisible goods. However, these property rights may be compromised by a tiered system. Define a tiered group of agents as a partition of N into L ordered sets, $N_1, ..., N_L$. The below axioms of tiered social justice are natural to replace individual rationality and envy-freeness in the traditional setting.

An allocation is said to be tiered envy-free if for all $i \in N_l$, all $j \in N_{l+1}, ..., N_L$, $x_i R_i x_j$. The motivation for this axiom is that we want to ensure that members of higher tiers receive better goods than those allocated to lower tiers, according to their preference ranking. This is the privilege curried by their high rank. However, we also wish to guarantee some base level of happiness for lower-ranked agents. So, we ensure that if we allocate the items respecting tiered envy-freeness that we also preserve some minimal level of utility for the lower-ranked agents. An allocation is said to be tiered individually rational if for each agent $i \in N_l$, either $x_i R_i E_0(i)$ or $x_i R_i E_0(j)$ for some $j \in N_1, ..., N_{l-1}$.

Note that tiered envy-freeness only pertains to an agent's relationship to those in the tiers below him, not to those in his tier.

3 The Set of Hierarchical Exchange Functions

Papai [4] characterizes the set of strategyproof, Pareto consistent, and nonbossy social choice functions for the assignment problem. This set is called hierarchical exchange functions, as they are the result of an iterative procedure where agents exchange objects from their hierarchically determined endowment sets. Note that, while serial dictatorships are hierarchical exchange functions, most hierarchical exchange functions are not serially dictatorial.

A typical hierarchical exchange function will behave as follows. All agents begin with an initial endowment, which may be of any size less than or equal to $|K|$. If any agent's top ranked good is in his endowment set, it is allocated to him. If any set of agents form a cycle, where each agent's top ranked object is in the endowment set of the next, the agents all trade. The goods in the endowment sets of those who have traded are then inherited by those still left, and the process repeats itself until all agents have receive their allocation.

Under assumptions in section 2.1 on the set of goods, agents, and their preferences, hierarchical exchange functions are the only mechanisms which are Pareto consistent, nonbossy, and strategyproof. The following rules constitute a hierarchical exchange function.[2] Define these rules iteratively and let $T_1(i, P) = top[P_i, K]$ be agent i's initial top choice from the set K, and let $E_1(i) \subseteq K$ be agent i's initial endowment. The initial endowment rule is for all $i \in N$, there exists an $E_1(i) \subseteq K$ such that $\cup_{i \in N} E_1(i) = K$,

[2]While Papai's characterization includes a variable $C_t(i, P)$, a "conditional endowment," it will become clear when we discuss tiered exchange functions that all hierarchical exchange functions which satisfy the axioms of tiered social justice will be such that $C_t(i, P) = \emptyset$ for all i and t. Therefore, for simplicity, we will only need to describe the hierarchical exchange functions for which $C_t(i, P) = \emptyset$.

and $\forall i, j \in N$, $E_1(i) \cap E_1(j) = \emptyset$. That is, each good is initially endowed to exactly one agent. Let

$$\begin{aligned} S_1(P) &= \{i \in N | T_1(i,P) \in E_1(i,P) \text{ or} \\ &\quad \text{there exists a sequence } j_1, ..., j_c \text{ such that } T_1(i,P) \in E_1(j_1,P), \\ &\quad T_1(j_i,P) \in E_1(j_{i+1},P), ..., T_1(j_{c-1},P) \in E_1(j_c,P), T_1(j_c,P) \in E_1(i,P)\} \end{aligned}$$

be the initial cycle. In other words, if there exists a sequence of agents such that each agents' top choice is in the endowment set of another (or his own), beginning and ending with the same agent, a cycle is formed and the agents trade. Let $F_1(P) = \{f_i(P) | i \in S_1(P)\}$ be the set of goods allocated in the initial cycle, such that if $i \in S_1(P), x_i \in F_1(P)$.

Defined iteratively, let $R_t(P) = \cup_{t=1}^{t} S_t(P)$ be the set of agents in cycles in the first t rounds and $H_t(P) = \cup_{t=1}^{t} F_t(P)$ be the set of goods allocated in the first t rounds, such that if $i \in R_t(P)$, then $x_i \in H_t(P)$. Let $T_t(i,P) = top[P_i, K \backslash H_{t-1}(P)]$ be agent i's top choice of the items remaining at time t and $E_t(i,P) \subseteq \{K \backslash H_{t-1}(P)\}$ is agent i's endowment at time t. Let a cycle at time period t be defined as

$$\begin{aligned} S_t(P) &= \{i \in N \backslash R_{t-1}(P) | T_t(i,P) \in E_t(i,P) \text{ or} \\ &\quad \text{there exists a sequence } j_1, ..., j_c \text{ such that } T_t(i,P) \in E_t(j_1,P), \\ &\quad T_t(j_i,P) \in E_t(j_{i+1},P), ..., T_t(j_{c-1},P) \in E_t(j_c,P), T_t(j_c,P) \in E_t(i,P)\} \end{aligned}$$

which can be interpreted analogously to the initial cycle.

Let T be last round, that is T is the first round for which $i \in R_T(P)$, for all $i \in N$. The hierarchical endowment rule is for all $t \in [2, T]$, all $P \in P$, and all $i \in N \backslash R_{t-1}(P)$, there exists an $E_t(i,P) \subseteq \{K \backslash H_{t-1}(P)\}$ such that $\cup_{i \in N \backslash R_{t-1}(P)} E_t(i,P) = K \backslash H_{t-1}(P)$ and for all $i, j \in N \backslash R_{t-1}(P)$, $E_t(i,P) \cap E_t(j,P) = \emptyset$. So, at every time period t, each unallocated good is endowed to exactly one remaining agent.

All hierarchical exchange functions have an award rule, such that for all $P \in P$, and all $i \in N, \forall t \in [1, T]$, $i \in S_t(P)$ implies $f_i(P) = T_t(i,P)$. If an agent trades in time t, he is allocated his most preferred good of those not allocated in the first $t-1$

rounds. Additionally, we have an awarded objects rule, such that for all $P \in P$, and all $t \in [1,T], F_t(P) = \{f_i(P) | i \in S_t(P)\}$.

Hierarchical exchange functions also place bounds on inheritance and property rights. We have an inheritance rule. Let $P, P' \in P$ such that $\exists t, t' \quad T$ with $S_t(P) = S_{t'}(P')$ and for all $i \in R_t(P), f_i(P) = f_i(P')$. Then, for all $i \in N \backslash R_t(P), E_{t+1}(i, P) = E_{t'+1}(i, P)$. Informally, if cycles don't change under two different preference profiles, then neither should the endowments. Finally, we have an assurance rule. Let $P \in P, i \in N, a \in K, t \quad T$, such that $a \in E_t(i, P)$ and $i \in N \backslash R_t(P)$, then $a \in E_{t+1}(i, P)$. So, if an agent i is endowed with a good a in time period t, and i does not trade in t, then i is endowed with a at time $t + 1$.

These rules describe the class of hierarchical exchange functions, which Papai [4] proves is a complete characterization of the set of strategyproof, Pareto consistent, and nonbossy social choice functions for the assignment problem.

3.1 Example of a Hierarchical Exchange Function

Imagine we have the following situation:

$$
\begin{aligned}
N &= \{1,2,3,4,5,6,7,8,9\} \\
K &= \{a,b,c,d,e,f,g,h,i,j\}
\end{aligned}
$$

Preferences for the nine agents are:

$$
\begin{aligned}
P_1 &= (d,e,b,c,a,g,h,f,i,j), P_2 = (d,e,j,a,b,c,f,g,h,i), P_3 = (b,j,i,h,g,f,e,d,c,a) \\
P_4 &= (a,b,c,d,e,f,g,h,i,j), P_5 = (h,g,i,j,a,c,b,f,e,d), P_6 = (f,g,h,i,j,a,b,c,d,e) \\
P_7 &= (e,c,a,b,d,j,i,h,g,f), P_8 = (i,j,h,g,f,d,e,b,c,a), P_9 = (h,g,j,i,a,b,c,d,e,f)
\end{aligned}
$$

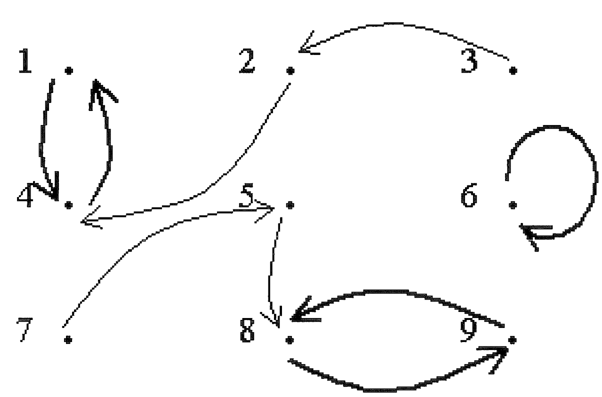
Which means that the following list represents each agents' initial top choice:

$$\begin{aligned} T_1(1,P) &= d, T_1(2,P) = d, T_1(3,P) = b \\ T_1(4,P) &= a, T_1(5,P) = h, T_1(6,P) = f \\ T_1(7,P) &= e, T_1(8,P) = i, T_1(9,P) = h \end{aligned}$$

Initial endowments are as follows:

$$\begin{aligned} E_1(1) &= \{a,j\}, E_1(2) = \{b\}, E_1(3) = \{c\} \\ E_1(4) &= \{d\}, E_1(5) = \{e\}, E_1(6) = \{f\} \\ E_1(7) &= \{g\}, E_1(8) = \{h\}, E_1(9) = \{i\} \end{aligned}$$

The diagram below represents the declarations in the first round, with each agent "pointing" to the agent who is currently endowed with his most preferred good.



Round 1 – Each agent "points" to the agent endowed with his most preferred good (bold arrows indicate trades which will occur this round)

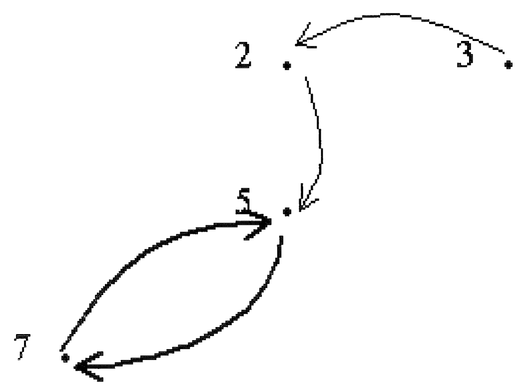
So, the first cycle is $S_1(P) = \{1,4,6,8,9\}$ with the corresponding award rule yielding $F_1(P) = \{d,a,f,i,h\}$. Now, we only have four agents remaining, $N \backslash S_1(P) = \{2,3,5,7\}$ and five items remaining, $K \backslash F_1(P) = \{b,c,e,g,j\}$. The second round endowments are:

$$E_2(2) = \{b,j\}, E_2(3) = \{c\}, E_2(5) = \{e\}, E_2(7) = \{g\}$$

The top choices of the four remaining agents over the five remaining goods are:

$$T_2(2,P)=e, T_2(3,P)=b, T_2(5,P)=g, T_2(7,P)=e$$

This yields this diagram for round two.



Round 2

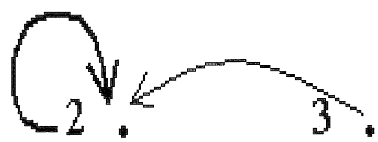
The second cycle is $S_2(P)=\{5,7\}$ and the corresponding awarded goods are $F_2(P)=\{g,e\}$. So, the set of all agents who have received their allocations is $R_2(P)=\{1,4,5,6,7,8,9\}$ and the set of all goods that have been allocated is $H_2(P)=\{d,a,g,f,e,i,h\}$. Now, we have two remaining agents $N\backslash R_2(P)=\{2,3\}$ and three remaining goods, $K\backslash H_2(P)=\{b,c,j\}$. The third round endowments are:

$$E_2(2)=\{b,j\}, E_2(3)=\{c\}$$

The top choices of the two remaining agents are:

$$T_2(2,P)=j, T_2(3,P)=b$$

So, we have the following for round three, where the third cycle is $S_3(P)=\{2\}$ and $F_3(P)=\{j\}$, $R_3(p)=\{1,2,4,5,6,7,8,9\}$ and $H_3(P)=\{d,j,a,g,f,e,i,h\}$.



Round 3

Now, agent three receives $E_3(P) = \{b, c\}$ and "points" to himself. So, $S_4(P) = \{3\}, F_4(P) = \{b\}$. So, all agents are members of $R_4(P) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $H_4(P) = x = \{d, j, b, a, g, f, e, i, h\}$ and good c remains unallocated.

Round 4

4 The Set of Tiered Exchange Functions

While hierarchical exchange functions are "nice," in the sense that they are Pareto consistent, nonbossy, and strategyproof, there are hierarchical exchange functions that do not yield allocations that satisfy the axioms of tiered social justice. To illustrate the need for new rules, let's reexamine the above example. Now, imagine that $N_1 = \{1, 2, 3\}$, $N_2 = \{4, 5, 6\}$, and $N_3 = \{7, 8, 9\}$ and recall that in our example the hierarchical exchange function yielded $x = \{d, j, c, a, g, f, e, i, h\}$. Note that in this allocation, $x_7 P_2 x_2$ and $x_7 P_5 x_5$, which violates tiered envy-freeness. That is, there is an assistant professor (tier 3) who has an office that both a full professor (tier 1) and an associate professor (tier 2) would like to have. Now, that doesn't seem fair, does it? We would like a mechanism to ensure that agents 1, 2 and 3 have the right of first refusal for all goods, before agents 4, 5 and 6 get to choose (and likewise for the third tier). On the other hand, agent 7, who somehow managed to have such a wonderful good to begin with, should be compensated in some reasonable way. We now characterize a class of mechanisms that ensure these properties hold.

There are many environments in which we have such a tiered set of agents. The set of agents, N, is partitioned into L ordered sets $N_1, ..., N_L$. In this environment, we would like any allocation to be such that tiered envy-freeness and tiered individual rationality hold. Additionally, the properties of strategyproofness, Pareto consistency, and nonbossiness remain attractive, and so we preserve these. Therefore, any set of mechanisms satisfying our criteria will be a subset of hierarchical exchange functions.

In order for a mechanism to implement tiered envy-free and tiered individually rational allocations for all $P \in P$, we must impose restrictions on the structure of the endowments of the hierarchical exchange functions. Tiered envy freeness implies that all $j \in N_1, ..., N_{l-1}$ must have the right of first refusal on $E_0(i)$, for all $i \in N_l$, before it becomes available to i. However, tiered individual rationality means that if $E_0(i)$ is taken by some j, i should be compensated in some reasonable manner. Since $E_0(j)$ was generally the result of an earlier iteration of this mechanism, it is a reasonable compensation. And, if these $j \in N_1, ..., N_{l-1}$ pass on $E_0(i)$, it is the property of i until he trades. That is, $E_0(i)$ only becomes available again in or after $S_i(P)$, the cycle in which i trades and receives x_i.

To ensure that tiered envy-freeness and tiered individual rationality are satisfied under all admissible circumstances, we must restrict the rules of the hierarchical exchange function. These restrictions are imposed entirely upon the endowment structure of the function. They must be altered as follows, yielding the tiered exchange function.

The first difference between the rules satisfied by hierarchical and tiered exchange functions is that, by imposing tiered individual rationality, we have set conditional endowments equal to the empty set. All members of N have some status quo allocation, $E_0(j)$, as described earlier. Since all agents receive a status quo endowment, no agent will ever receive a conditional endowment. Therefore, we can concern ourselves solely with the subset of hierarchical exchange functions where $C_t(i, P) = \emptyset$ for all i, t. Recall that in hierarchical exchange functions, the initial endowment rule is that for all $i \in N$, there exists $E_1(i) \subseteq K$ such that $\cup_{ieN} E_1(i) = K$, and for all $i, j \in N$, $E_1(i) \cap E_1(j) = \emptyset$. We wish to ensure that members of lower tiers do not receive the rights to their endowment until all members of higher tiers have given up their claim, in order to satisfy tiered envy-freeness. So, we add the following restrictions and create a tiered initial endowment rule for tier 1.

$$E_0(i) \subseteq E_1(i) \text{ for all } i \in N_1, \text{and } E_1(i) = \emptyset \text{ for all } i \in N_2, ..., N_L$$

However, we also wish to ensure that all members have a some sort of right to their status quo endowment, as we have to ensure that tiered individual rationality is preserved. So, we create a tiered initial endowment rule for tier l. Let t_l be the time period in which the last member of tier N_l trades. Then, for all $i \in N_l$,there exists an $E_{t_{l-1}+1}(i) \subseteq K \backslash H_{t_{l-1}}(P)$ such that

$$\text{if } E_0(i) \neq x_j \text{ for all } j \in N_1, ..., N_{l-1}, E_0(i) \subseteq E_{t_{l-1}+1}(i)$$

However, if there exists a $j \in N_1, ..., N_{l-1}$ such that $E_0(i) = x_j$, then there exists a sequence $j_1, ..., j_c \in N_1, ..., N_{l-1}$ such that

$$E_0(i) = x_j, E_0(j) = x_{j_1}, E_0(j_1) = x_{j_2}, ..., E_0(j_{c-1}) = x_{j_c} \text{with } E_0(j_c) \in E_{t_{l-1}+1}(i)$$

In other words, if the status quo endowment of agent i in tier l has not become the allocation of some higher ranked agent, then once it has come time for agent i to receive a non-empty endowment, he will receive his status quo as part of his endowment set. However, if agent i's status quo endowment is allocated to a more highly ranked individual, then he will receive the "leftover" status quo good at the end of the trading above him. Additionally, in the same spirit as the tiered initial endowment rule for tier 1, $E_{t_{l-1}+1} = \emptyset$ for all $i \in N_{l+1}, ..., N_L$. The remaining constraints are similar to those for the hierarchical endowment rule. That is, $\cup_{i \in N} E_{t_{l-1}+1}(i) = K \backslash H_{t_{l-1}+1}(P)$, and for all $i, j \in N_i$, $E_{t_{i-1}+1}(i) \cap E_{t_{i-1}+1}(j) = \emptyset$.

4.1 Example of a Tiered Exchange Function

Let us return to our example. Imagine that $N_1 = \{1,2,3\}$, $N_2 = \{4,5,6\}$, and $N_3 = \{7,8,9\}$ and recall that in our earlier example, the hierarchical exchange function yielded $x = \{d,j,c,a,g,f,e,i,h\}$ and that in this allocation, $x_7 P_2 x_2$ and $x_7 P_5 x_5$, which violates tiered envy-freeness. However, if we use a tiered exchange function, we would avoid this problem. Consider the following example, recalling the following sets of agents, goods, and preferences from our earlier example:

$$
\begin{aligned}
N &= \{1,2,3,4,5,6,7,8,9\} \\
K &= \{a,b,c,d,e,f,g,h,i,j\} \\
P_1 &= (d,e,b,c,a,g,h,f,i,j), P_2 = (d,e,j,a,b,c,f,g,h,i), P_3 = (b,j,i,h,g,f,e,d,c,a) \\
P_4 &= (a,b,c,d,e,f,g,h,i,j), P_5 = (h,g,i,j,a,c,b,f,e,d), P_6 = (f,g,h,i,j,a,b,c,d,e) \\
P_7 &= (e,c,a,b,d,j,i,h,g,f), P_8 = (i,j,h,g,f,d,e,b,c,a), P_9 = (h,g,j,i,a,b,c,d,e,f)
\end{aligned}
$$

Which means that the following list represents each agents' initial top choice:

$$
\begin{aligned}
T_1(1,P) &= d, T_1(2,P) = d, T_1(3,P) = b \\
T_1(4,P) &= a, T_1(5,P) = h, T_1(6,P) = f \\
T_1(7,P) &= e, T_1(8,P) = i, T_1(9,P) = h
\end{aligned}
$$

Status quo endowments are as follows:

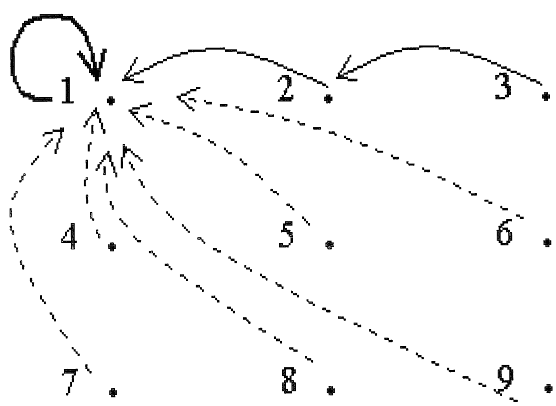
$$
\begin{aligned}
E_0(1) &= \{a\}, E_0(2) = \{b\}, E_0(3) = \{c\} \\
E_0(4) &= \{d\}, E_0(5) = \{e\}, E_0(6) = \{f\} \\
E_0(7) &= \{g\}, E_0(8) = \{h\}, E_0(9) = \{i\}
\end{aligned}
$$

However, initial endowments are rearranged to satisfy the initial endowment rule for tier 1:

$$
E_1(1) = \{a,d,e,f,g,h,i,j\}, E_1(2) = \{b\}, E_1(3) = \{c\}
$$

$$E_1(4) = E_1(5) = E_1(6) = E_1(7) = E_1(8) = E_1(9) = \emptyset$$

Below is the diagram which represents the declarations in the first round, with each agent "pointing" to the agent who is currently endowed with his most preferred object. Note that the agents in the second and third tiers may point, but they will never trade before all the members of the first tier pass on their endowments.



Round 1 – Dashed lines emanate from those agents without endowments, and who therefore cannot trade in this round

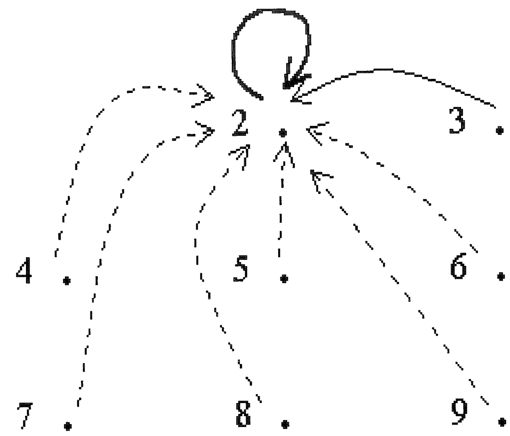
So, the first cycle is $S_1(P) = \{1\}$ with the corresponding award rule yielding $F_1(P) = \{d\}$. Now, we have eight agents remaining, $N \backslash S_1(P) = \{2,3,4,5,6,7,8,9\}$ and nine items remaining, $K \backslash F_1(P) = \{a,b,c,e,f,g,h,i,j\}$. The second round endowments are:

$$\begin{aligned} E_2(2) &= \{a,b,c,e,f,g,h,i,j\}, E_2(3) = \{c\} \\ E_2(4) &= E_2(5) = E_2(6) = E_2(7) = E_2(8) = E_2(9) = \emptyset \end{aligned}$$

The top choices of the remaining agents over the remaining goods are:

$$\begin{aligned} T_2(2,P) &= e, T_2(3,P) = b \\ T_2(4,P) &= a, T_2(5,P) = h, T_2(6,P) = f \\ T_2(7,P) &= e, T_2(8,P) = i, T_2(9,P) = h \end{aligned}$$

This yields the following diagram for round two.



Round 2

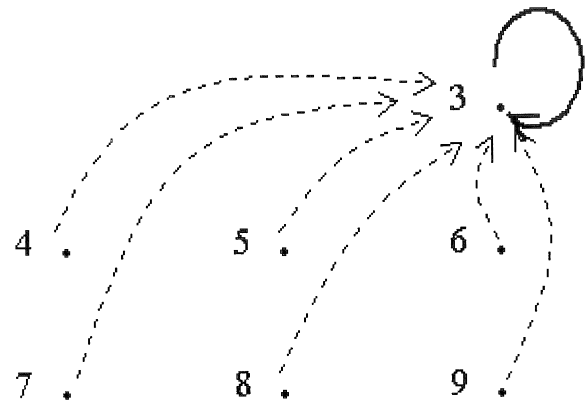
The second cycle is $S_2(P) = \{2\}$ and the corresponding awarded goods are $F_2(P) = \{e\}$. So, the set of all agents who have received their allocations is $R_2(P) = \{1, 2\}$ and the set of all goods that have been allocated is $H_2(P) = \{d, e\}$. Now, we have remaining agents $N \backslash R_2(P) = \{3, 4, 5, 6, 7, 8, 9\}$ and remaining goods $K \backslash H_2(P) = \{a, b, c, f, g, h, i, j\}$. The third round endowments are:

$$
\begin{aligned}
E_3(3) &= \{a, b, c, f, g, h, i, j\} \\
E_3(4) &= E_3(5) = E_3(6) = E_3(7) = E_3(8) = E_3(9) = \emptyset
\end{aligned}
$$

The top choices of the remaining agents are:

$$
\begin{aligned}
T_3(3, P) &= b \\
T_3(4, P) &= a, T_3(5, P) = h, T_3(6, P) = f \\
T_3(7, P) &= c, T_3(8, P) = i, T_3(9, P) = h
\end{aligned}
$$

This yields the following figure for round three.



Round 3

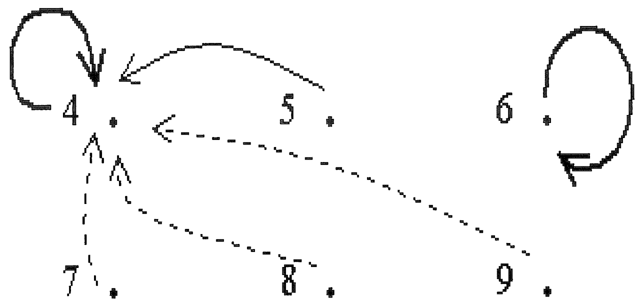
The third cycle is $S_3(P) = \{3\}$ and $F_3(P) = \{b\}$, with $R_3(P) = \{1, 2, 3\}$ and $H_3(P) = \{d, e, b\}$. Now, we have remaining agents $N \backslash R_3(P) = \{4, 5, 6, 7, 8, 9\}$ and remaining goods, $K \backslash H_3(P) = \{a, c, f, g, h, i, j\}$. The fourth round endowments, which obey the initial endowment rule for tier l, are:

$$\begin{aligned} E_4(4) &= \{a, g, h, i, j\}, E_4(5) = \{c\}, E_4(6) = \{f\} \\ E_4(7) &= E_4(8) = E_4(9) = \emptyset \end{aligned}$$

The top choices of the remaining agents are:

$$\begin{aligned} T_4(4, P) &= a, T_4(5, P) = h, T_4(6, P) = f \\ T_4(7, P) &= c, T_4(8, P) = i, T_4(9, P) = h \end{aligned}$$

So, we have the below diagram for round four.



Round 4

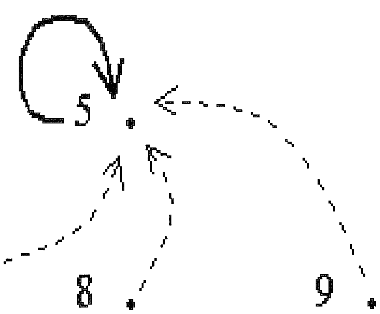
The fourth cycle is $S_4(P) = \{4,6\}$ and $F_4(P) = \{a, f\}$, $R_4(P) = \{1,2,3,4,6\}$, $H_4(P) = \{d, e, b, a, f\}$. Now, fifth round endowments are:

$$\begin{aligned} E_5(5) &= \{c, g, h, i, j\} \\ E_5(7) &= E_5(8) = E_5(9) = \emptyset \end{aligned}$$

The top choices of the remaining agents are:

$$\begin{aligned} T_5(5, P) &= h \\ T_5(7, P) &= c, T_5(8, P) = i, T_5(9, P) = h \end{aligned}$$

Which yields:



Round 5

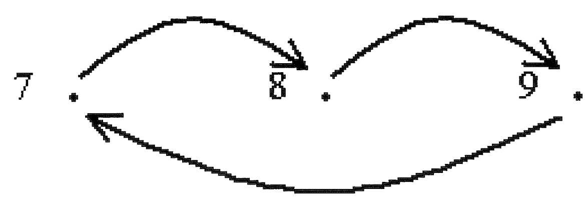
The fifth cycle is $S_5(P) = \{5\}$, with $F_5(P) = \{h\}, R_5(P) = \{1, 2, 3, 4, 5, 6\}, H_5(P) = \{d, e, b, a, h, f\}$. Now, sixth round endowments are:

$$E_6(7) = \{g, j\}, E_6(8) = \{c\}, E_6(9) = \{i\}$$

The top choices of the remaining agents are:

$$T_6(7, P) = c, T_6(8, P) = i, T_6(9, P) = g$$

This gives us the following diagram for the sixth and final round.



Round 6

So, the sixth cycle is $S_6(P) = \{7, 8, 9\}$ with $F_6(P) = \{c, i, g\}, R_4(P) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $H_4(P) = x = \{d, e, b, a, h, f, c, i, g\}$. Note that this allocation satisfies both tiered individual rationality and tiered envy freeness.

4.2 Properties of Tiered Exchange Functions

Because tiered exchange functions are a specific class of hierarchical exchange functions, we know that they are strategyproof, non-bossy, and Pareto consistent. Additionally, we can show that since $C_t(i, P) = \emptyset$, these functions are coalitionally strategyproof. Furthermore, this holds for all hierarchical exchange functions where $C_t(i, P) = \emptyset$ for all i, t. In order to do this, we first establish the following lemma, which states that if an agent i prefers $f_i(P)$ to $f_i(P')$, then there must be another agent j who, under P', prefers an item endowed to a member who trades after i under P. Both the lemma and the proof are adaptations of Bird [1].

1. **Lemma 1** *Let $x = (x_1, ..., x_n)$ be the allocation when the report is $P = (P_1, ..., P_n)$, and let $x' = (x'_1, ..., x'_n)$ be the allocation when the report is $P' = (P'_1, ..., P'_n)$. Then, if there exists an $i \in N$ such that $x'_i P_i x_i$ and $i \in S_k(P)$, then there exists a j such that $j \in R_{k-1}(P)$ and an $h \in N \backslash R_{k-1}(P)$ such that $w_h P'_j x_j, w_h \in E_k(h, P')$.*

Proof. Assume not. Then $x_m P'_m w_n, w_n \in E_k(n, P')$ for all $m \in R_{k-1}(P)$ and all $n \in N \backslash R_{k-1}(P)$. Note that $x_m = w_i$ for some $i \in R_{k-1}(P)$. So, $R_{k-1}(P) = R_{k-1}(P')$. Since $x'_i P_i x_i$, for $i \in S_k(P)$, x'_i must be taken before k under P. So, $x'_i = w_j$ for some $j \in R_{k-1}(P)$. Since i get x'_i under P', i and j are in the same period under P', so $i \in R_{k-1}(P')$. But $R_{k-1}(P) = R_{k-1}(P')$, and $i \notin R_{k-1}(P)$. Contradiction. ■

Using this lemma, we establish that tiered exchange functions are coalitionally strategyproof. This is proven by choosing a member of a coalition who is the first to lie. Then, we show that the only way he can benefit from his lie is if he isn't the first to do so.

Theorem 2 *Tiered exchange functions are coalitionally strategyproof. That is, for all $C \subseteq N$, all $P \in \mathcal{P}$, and all $P'_C \in \mathcal{P}_C$, there exists an $i \in C$ such that $x_i(P) P_i x_i(P'_C, P_{-C})$.*

Proof. Choose $i \in C$ such that $i \in S_k(P)$ and $h \in N \backslash R_t(P)$, $\forall h \in C, h \neq i$. If $x'_i P_i x_i$, $x'_i = x_j$ for some $j \in R_{k-1}(P)$, by the above lemma. So, in order for x'_i to be available to i under P', j must be lying under P'. So, i can't be the first member of C to trade. ■

Additionally, tiered exchange functions will implement allocations which satisfy tiered individual rationality and tiered envy-freeness, for all admissible preference profiles, $P \in P$.

Theorem 3 *Tiered exchange functions select allocations which satisfy tiered individual rationality and tiered envy-freeness.*

1. Tiered Individual Rationality

   Proof. By construction

2. Tiered Envy-Freeness:

   Proof. Order players such that if i is removed in an earlier round than j, $i < j$, and if they are removed in the same round, order arbitrarily. With this ordering, $x_i P_i x_j$. This makes the tiered exchange function a sequential choice function. By definition of the mechanism, $S_i(P) < S_j(P)$ (in a slight abuse of notation, call $S_i(P)$ the round in which i trades, and $S_j(P)$ the round in which j trades) for all $i \in N_i$ and all $j \in N_{i+1}, ..., N_L$. Therefore, $x_i P_i x_j$ for all $i \in N_l$ and all $j \in N_{l+1}, ..., N_L$ ■

Moreover, the following theorem shows that if f is a strategyproof, nonbossy social choice function which implements Pareto efficient, tiered envy free and tiered individually rational allocations for all admissible preference profiles, then f must be a tiered exchange function. By Papai [4], strategyproofness, nonbossiness, and Pareto consistency imply that we must be considering only hierarchical exchange functions. The theorem below demonstrates that the only hierarchical exchange functions which implement tiered envy free and tiered individually rational allocations for all admissible preference profiles are tiered exchange functions.

Theorem 4 *If a strategyproof, nonbossy SCF f is such that $f(P) = x$ where x is tiered individually rational, tiered envy free, and Pareto efficient for all $P \in \mathcal{P}$, then f must be a tiered exchange function.*

Method: Suppose f is a hierarchical exchange function which is not a tiered exchange function. Suppose further that $f(P) = x$, where x is a tiered-envy free, tiered individually rational, and Pareto efficient allocation. Then there exists a $P' \in P$ such that $f(P') = x'$ violates tiered individual rationality, tiered envy-freeness, or both.

Proof. This proof will proceed in parts, violating one aspect of tiered exchange functions at a time:

Case 1 The hierarchical exchange function violates the initial endowment rule

This means that instead of

$$E_0(i) \subseteq E_1(i) \text{ for all } i \in N_1, \text{and } E_1(i) = \emptyset \text{ for all } i \in N_2, ..., N_L$$

the rule reads:

$$E_0(i) \subseteq E_1(i) \text{ for all } i \in N_1, \text{and } E_1(i) \neq \emptyset \text{ for some } i \in N_2, ..., N_L \tag{1}$$

or:

$$E_0(i) \nsubseteq E_1(i) \text{ for all } i \in N_1, \text{and } E_1(i) = \emptyset \text{ for all } i \in N_2, ..., N_L \tag{2}$$

If (1) holds: There exists an $i \in N_2, ..., N_L$ such that $E_1(i) \neq \emptyset$. Let $x_{1i} \in E_1(i)$. Suppose that $T_1(i, P^{'}) = x_{1i}$ and that there exists an $i \in N_1$ such that $T_1(j, P^{'}) = x_{1i}$. By the awarded agent rule, $i \in S_1(P^{'})$ and by the award rule, $f_i(P^{'}) = T_1(i, P^{'}) = x_{1i}$. But $T_1(j, P^{'}) = x_{1i}$. So, the best agent j can get is his second choice and regardless of his final allocation, x_j, $x_i P_j^{'} x_j$. This violates tiered envy-freeness.

If (2) holds: Suppose $T_1(i, P^{'}) = E_0(i)$. Suppose further that $x_j = E_0(j), \forall j \in N_1$. Suppose that $E_0(i) \in E_1(h), h \in N_1$ and that $T_1(h, P^{'}) = E_0(i)$. Therefore, $h \in S_1(P^{'})$, and $f_h(P^{'}) = T_1(h, P^{'}) = E_0(i)$. This violates tiered individual rationality.

Case 2 The hierarchical exchange function violates the initial endowment rule for tier l, which reads:

Let t_i be the time period in which the last member of tier N_l trades. Then, for all $i \in N_l$,there exists an $E_{t_{l-1}+1}(i) \subseteq K \backslash H_{t_{l-1}}(P)$ such that

$$\text{if } E_o(i) \neq x_j \text{ for all } j \in N_1, ..., N_{l-1},\ E_0\,(i) \subseteq E_{t_{l-1}+1}(i) \tag{3}$$

However, if there exists a $j \in N_1, ..., N_{l-1}$such that $E_0(i) = x_j$, then there exists a sequence $j_1, ..., j_c \in N_1, ..., N_{l-1}$ such that

$$E_0(i) = x_j, E_0(j) = x_{j_1}, E_0(j_1) = x_{j_2}, ..., E_0(j_{c-1}) = x_{j_c} \text{with } E_0(j_c) \in E_{_{t_{l-1}+1}}(i) \tag{4}$$

1. (a) Suppose The hierarchical exchange function violates 3: So, $E_0(i) \neq x_j$ and $E_0(i) \nsubseteq E_{t_{i-1}+1}(i)$. Suppose $T_{t_{i-1}+1}(i, P^{'}) = E_0(i)$. Suppose further that $x_j = E_0(j)$,for all $j \in N_1, ..., N_{i-1}$. Suppose that $E_0(i) \in E_{t_{i-1}+1}(h), h \in N_i$ and that $T_{_{t_{i-1}+1}}(h, P^{'}) = E_0(i)$. Therefore, $h \in S_{_{t_{i-1}+1}}(P^{'})$, and $f_h(P^{'}) = T_{_{t_{i-1}+1}}(h, P^{'}) = E_0(i)$. This violates tiered individual rationality.

(b) Suppose The hierarchical exchange function violates 4: So, $E_0(i) = x_j$ for some $j \in N_1, ..., N_{l-1}$ and there does not exist a sequence $j_1, ..., j_c \in N_1, ..., N_{l-1}$ such that $E_0(i) = x_j, E_0(j) = x_{j_1}, E_0(j_1) = x_{j_2}, ..., E_0(j_{c-1}) = x_{j_c}$ such that $E_0(j_c) \in E_{t_{l-1}+1}(i)$. So, $E_0(j_c) \notin E_{t_{i-1}+1}(i)$. Suppose that P' is such that $\forall g \neq j, g \in N_1, ..., N_{i-1}, x_g = E_0(g)$. So, all $E_0(g)$ are taken by time $t_{l-1}+1$. Suppose further that $E_0(j) \in E_{t_{l-1}+1}(h), h \in N_i$ and that $T_{t_{l-1}+1}(h, P') = E_0(i)$. Therefore, $h \in S_{t_{l-1}+1}(P')$, and $f_h(P') = T_{t_{l-1}+1}(h, P') = E_0(i)$. This violates tiered individual rationality.

Case 3: The hierarchical exchange function is such that for some agent in some time period $C_t(i, P) \neq \emptyset$.

1. Suppose there exists a j such that $T_1(j, P) = C_t(i, P)$. Suppose further that if i were allowed to keep his conditional endowment (were it not conditional) then he would. Furthermore, suppose that his second top choice were in $E_1(j, P)$. Then, i and j would trade, and we violate tiered individual rationality. ■

We can also show that for any x satisfying tiered individual rationality, tiered envy-freeness and Pareto efficiency, there exists a tiered exchange function which implements it.

Theorem 5 *Given $f : P \to X$ such that $f(P)$ is tiered envy free, tiered individual rationality, and Pareto efficient, then there exists a Pareto consistent, nonbossy, strategyproof tiered exchange function, g, such that $g(P) = f(P)$ for all P.*

Proof. Let $x = (x_{11}, ..., x_{n_1 1}, x_{12}, ..., x_{n_2 2}, ..., x_{n_L L})$ be an allocation satisfying tiered envy-freeness, tiered individual rationality, and Pareto consistency. The following algorithm implements it in a strategyproof, nonbossy manner.

1. In round one, allocate all of the items in x to members of tier one, as follows:

   $E_1(i) = x_{i1}$ for all $i \in \{11, ..., (n_1 - 1)1\}$

   $E_1(i) = \{x_{i1}, x_{12}, ..., x_{n_2 2}, ..., x_{n_L L}\}, i = n_1 1$

- Since x satisfies tiered envy-freeness, $x_{i1}P_{i1}x_{ij}\forall j > 1$. Therefore, there does not exist a $S_t(P)$ such that $x_{i1} \in \{E_1(n_1 1)\backslash x_{n_1 1}\}$. That is, there will never be a cycle in which a member of tier one trades for an element of $\{E_1(n_1 1)\backslash x_{n_1 1}\}$
- Since x is Pareto efficient, there does not exist a $x' \in X$ such that $x_i' P_i x_i \forall i \in N$
- Therefore, the cycles formed by the members of tier one will all be such that $T_t(i, P) \in E_1(i, P)$

2. So, all members of tier one will be awarded $E_1(i)$. Member $n_1 1$ will be awarded $x_{n_1 1}$, and the rest of his endowment vector will be passed along to tier two in the following manner:

   $E_t(i) = x_{i2}$ for all $i \in \{12, ..., (n_2 - 1)2\}$

   $E_t(i) = \{x_{i2}, x_{13}, ..., x_{n_3 3}, ..., x_{n_L L}\}, i = n_2 2$

   - By the same argument as above, all allocations will be taken from the endowment sets, and we pass along the unallocated goods in an analogous manner until we get to tier L

3. $E_t(i) = x_{iL}$ for all $i \in N_L$

   - By the same argument as above, all allocations will be equal to the agents' endowment sets ■

There are two extreme cases of the tiered exchange function. Note that when $L = N$, we have every player in their own tier. In this case, the tiered exchange function is a serial dictatorship, where $i \in N_1$ is endowed with all $a \in K$ and chooses his most preferred object, then $j \in N_2$ is endowed with $K\backslash x_i$, and chooses his most preferred from this set, and so on. Additionally, when $L = 1$, the tiered exchange function behaves like a general hierarchical exchange function.

5 Conclusions

In this paper, we define new axioms of tiered social justice based upon the Platonic ideal. We then establish that there is a subclass of hierarchical exchange functions whose allocations satisfy these new axioms, namely tiered envy-freeness and tiered individual rationality. We also determine that when $C_t(i, P) = \emptyset$ for all i and t, that hierarchical exchange functions are coalitionally strategyproof. Furthermore, we demonstrate that given a social choice function, f, such that the resultant allocation is tiered envy free, tiered individually rational, and Pareto efficient, there exists a Pareto consistent, nonbossy, strategyproof tiered exchange function, g, such that the two functions are observationally equivalent for all admissible preference profiles.

In the near future, we hope to explore the impact of synergies on this class of mechanisms. That is, what if my preferences over the goods is a function of who is allocated each good? Additionally, we hope to explore the possibility of extending this class to environments in which agents may desire to be allocated more than one good.

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